MIXED 3–MANIFOLDS ARE VIRTUALLY SPECIAL

PIOTR PRZYTYCKI† AND DANIEL T. WISE‡

Abstract. Let M be a compact oriented irreducible 3–manifold which is neither a graph manifold nor a hyperbolic manifold. We prove that the fundamental group of M is virtually special.

1. Introduction

A compact connected oriented irreducible 3–manifold is *mixed* if it is not hyperbolic and not a graph manifold. A group is *special* if it is a subgroup of a right-angled Artin group. Our main result is the following.

Theorem 1.1. *Let M be a mixed 3–manifold. Then $\pi_1 M$ is virtually special.*

Corollary 1.2. *The fundamental group of a mixed 3–manifold is linear over $\mathbb{Z}$.*

As explained below, Theorem 1.1 has the following consequence for manifolds with toroidal boundary, possibly empty.

Corollary 1.3. *A mixed 3–manifold with toroidal boundary virtually fibers.*

An alternative definition of a special group is the following. A nonpositively curved cube complex X is *special* if its immersed hyperplanes do not self-intersect, are two-sided, do not directly self-osculate or interosculate (see Definition 4.1). A group G is *special* if it is the fundamental group of a *special* cube complex X. The equivalence with the previous definition was proved in [HW08, Thm 4.2].

Special groups are residually finite. Moreover, their subgroups that stabilise a hyperplane in the universal cover $\widetilde{X}$ of X are separable (see Corollary 4.8). For 3–manifold groups, separability of a subgroup corresponding to an immersed incompressible surface implies that in some finite cover of the manifold the surface lifts to an embedding. There are immersed incompressible surfaces in graph manifolds that do not lift to embeddings in a finite cover [RW98].

There are a variety of groups with the property that every finitely generated subgroup is separable — for instance, this was shown for free groups by M. Hall and for surface groups by Scott. A compact 3–manifold is *hyperbolic* if its interior is homeomorphic to a quotient of $\mathbb{H}^3$ (equivalently to the quotient of the interior of the convex hull of the limit set) by a geometrically finite Kleinian group. It was recently proved that hyperbolic 3–manifolds with an embedded geometrically finite incompressible surface have virtually special fundamental groups [Wis11]. If such a manifold has no boundary tori,

† Partially supported by MNiSW grant N201 012 32/0718 and the Foundation for Polish Science.
‡ Supported by NSERC.

the associated virtually special cube complex is compact, otherwise it might be *sparse*, see Section 2. Nevertheless, its fundamental group does contain a finite index subgroup that is $\pi_1 X$ with X compact special, though we will not use that fact. This implies separability for all geometrically finite subgroups [Wis11, Thm 16.23]. By the tameness [Ago04, CG06] and covering [Thu80, Can96] theorems all other finitely generated subgroups correspond to virtual fibers and hence they are separable as well. Very recently, Agol, Groves and Manning [Ago12, Thm 1.1] building on [Wis11] proved that the fundamental group of every closed hyperbolic 3–manifold is virtually compact special and hence all its finitely generated subgroups are separable. For more details, see the survey article [AFW12].

Another striking consequence of virtual specialness is virtual fibering. Since special groups are subgroups of right-angled Artin groups, they are subgroups of right-angled Coxeter groups as well [HW99, DJ00]. Agol proved that such groups are virtually residually finite rationally solvable (RFRS) [Ago08, Thm 2.2]. Then he proved ([Ago08, Thm 5.1]) that if the fundamental group of a compact connected oriented irreducible 3–manifold with toroidal boundary is RFRS, then it virtually fibers. In view of these results, every hyperbolic manifold with toroidal boundary virtually fibers [Ago12, Thm 9.2]. Similarly, our Theorem 1.1 yields Corollary 1.3.

The question of virtual specialness for graph manifolds was answered by Liu [Liu11, Thm 1.1], who proved that a graph manifold which is not Seifert fibered has virtually special fundamental group if and only if it admits a nonpositively curved Riemannian metric. Independently, and with an eye towards the results presented here, we proved virtual specialness for graph manifolds with nonempty boundary [PW11, Cor 1.3]. Note that graph manifolds with nonempty boundary carry a nonpositively curved metric by [Lee95, Thm 3.2]. Our Theorem 1.1 thus resolves the question of virtual specialness for arbitrary compact 3–manifold groups.

Corollary 1.4. *A compact aspherical* 3*–manifold has virtually special fundamental group if and only if it admits a Riemannian metric of non-positive curvature.*

Corollary 1.4 was conjectured by Liu [Liu11, Conj 1.3]. As discussed above, he proved the conjecture for graph manifolds while for hyperbolic manifolds this follows from [Wis11] and [Ago12]. All mixed manifolds admit a metric of nonpositive curvature [Lee95, Thm 3.3], [Bri01, Thm 4.3]. Hence Theorem 1.1 resolves Liu's conjecture in the remaining mixed case. However, the equivalence in Corollary 1.4 appears to be more circumstantial than a consequence of an intrinsic relationship between nonpositive curvature and virtual specialness: all manifolds in question except for certain particular closed graph manifolds have both of these features.

As a consequence of virtual specialness of mixed manifolds (Theorem 1.1), hyperbolic manifolds with boundary ([Wis11, Thm 14.29]), and graph manifolds with boundary ([PW11, Cor 1.3]) we have the following.

Corollary 1.5. *The fundamental group of any knot complement in S^3 has a faithful representation in* $\mathrm{SL}(n,\mathbb{Z})$ *for some n.*

Note that the existence of a non-abelian representation of any nontrivial knot complement group into SU(2) is a well-known result of Kronheimer–Mrowka [KM04].

The proof of Theorem 1.1 is divided into two steps, which is explained in Section 2 but recommended to skip at a first reading. The first step is Theorem 2.1 (Cubulation), which roughly states that in any mixed manifold there is a collection of surfaces sufficient for cubulation. We prove Theorem 2.1 in Section 3 by merging the surfaces used for cubulation of the graph manifold blocks and hyperbolic blocks. The second step is Theorem 2.4 (Specialisation), which states that the associated cube complex is virtually special. Since subcomplexes involved are not compact, in order to prove Theorem 2.4 we extend in Section 4 known separability results for compact special cube complexes to non-compact ones. We apply them in Section 5 to obtain cubical small-cancellation results for non-compact special cube complexes. This allows us to prove Theorem 2.4 in Section 6.

What the proof of Theorem 1.1 uses:

- Canonical completion and retraction Theorem 4.3 for special cube complexes [HW10].
- Criterion 2.3 for virtual specialness [HW10].
- Gitik–Minasyan double quasiconvex coset separability [Min06].
- Criterion for relative quasiconvexity [BW11].
- Relative cocompactness of cubulations of relatively hyperbolic groups [HW11].
- Proposition 3.2 which constructs virtually embedded surfaces in graph manifolds with boundary [PW11].
- Separability and double separability of embedded surfaces in graph manifolds [PW11].
- Special Quotient Theorem 6.1 for groups hyperbolic relative to virtually abelian subgroups [Wis11].
- Theorem 3.7 and Lemma 3.12 constructing geometrically finite surfaces in hyperbolic manifolds with boundary [Wis11].
- Virtual specialness of the sparse cube complex coming from these surfaces [Wis11].
- Main Theorem 5.1 of cubical small cancellation [Wis11].

Acknowledgements. We thank Stefan Friedl for his remarks and corrections.

2. Technical reduction to two steps

Let M be a compact connected oriented irreducible 3–manifold. We can also assume, by passing to a double cover, that M has no π_1–injective Klein bottles. Up to isotopy, M has a unique minimal collection of incompressible tori not parallel to ∂M, called *JSJ tori*, such that the complementary components called *blocks* are either atoroidal or else toroidal and Seifert fibered. If M is mixed, then it has at least one JSJ torus and one atoroidal block. By Thurston's hyperbolisation all atoroidal blocks are hyperbolic and we will denote them by M^h_k. The JSJ tori adjacent to hyperbolic blocks are *transitional*. The complementary components of the union of the hyperbolic blocks are graph manifolds with boundary and will be called *graph manifold blocks* and denoted by M^g_i. Each of their Seifert fibered blocks admits a unique Seifert fibration. If a transitional torus is adjacent on both of its sides to hyperbolic blocks, we replace it by two parallel tori (also called JSJ, and transitional) and add the product region $T \times I$ bounded by them as a graph manifold block to the family $\{M^g_i\}$. Similarly, for a boundary torus of M adjacent

to a hyperbolic block, we introduce its parallel copy in M (called JSJ, and transitional) and add the product region to $\{M_i^g\}$. We fix one of many Seifert fibrations on such M_i^g.

Unless stated otherwise all surfaces are embedded or immersed *properly*. Let $S \to M$ be an immersed surface in a 3–manifold M. Let $\widehat{M} \to M$ be a covering map. A map $\widehat{S} \to \widehat{M}$ which covers $S \to M$ and does not factor through another such map is its *elevation* (it is a *lift* when $\widehat{S} = S$). An aspherical oriented surface S is *immersed incompressible* if it is π_1–injective and its elevation to the universal cover $\widetilde{M}$ of M is an embedding. The surface S is *virtually embedded* if there is a finite cover $\widehat{M}$ of M with an embedded elevation of S. Note that by passing to a further cover one can assume $\widehat{M} \to M$ to be regular, and then by quotienting by $\pi_1 S$ one can assume $\widehat{S} = S$. Given a block B and an immersed surface $S \xrightarrow{\phi} M$ a *piece* is the restriction of ϕ to a component of $S \cap \phi^{-1}(B)$, which will be denoted simply by $S \cap B$.

We call the elevations of JSJ tori, boundary tori, and transitional tori of M to the universal cover $\widetilde{M}$ *JSJ planes*, *boundary planes*, and *transitional planes*, and keep the name *blocks* (hyperbolic, graph manifold, or Seifert fibered) for the elevations of blocks of M. Having specified a block $\widetilde{M}_o$ of $\widetilde{M}$ and a surface $\widetilde{S}_o$ in $\widetilde{M}_o$ we denote by $\mathcal{T}(\widetilde{S}_o)$ the set of JSJ and boundary planes in ∂M_o intersecting $\widetilde{S}_o$.

An *axis* for an element $g \in \pi_1 M$ acting on $\widetilde{M}$ is a g–invariant copy of $\mathbb{R}$ in $\widetilde{M}$. A *cut-surface* for g is an immersed incompressible surface $S \to M$ covered by $\widetilde{S} \subset \widetilde{M}$ such that there is an axis $\mathbb{R}$ for g satisfying $g^n(\widetilde{S}) \cap \mathbb{R} = \{n\}$ for all $n \in \mathbb{Z}$.

Theorem 2.1 (Cubulation)**.** *Let M be a mixed 3–manifold. There exists a finite family of immersed incompressible surfaces $\mathcal{S}$ in M, in general position, with the following properties:*

(1) For each element of $\pi_1 M$ there is a cut-surface in $\mathcal{S}$.
(2) All JSJ tori belong to $\mathcal{S}$.
(3) Each component of $S \cap M_i^g$ is virtually embedded in M_i^g for any $S \in \mathcal{S}$.
(4) Each component of $S \cap M_k^h$ is geometrically finite for any $S \in \mathcal{S}$..
(5) The family $\widetilde{\mathcal{S}}$ of surfaces in the universal cover $\widetilde{M}$ of M covering the surfaces in $\mathcal{S}$ satisfies the following Strong Separation property.

Definition 2.2. A family $\widetilde{\mathcal{S}}$ of surfaces in $\widetilde{M}$ satisfies the *Strong Separation* property if the following hold.

(a) For any $\widetilde{S}, \widetilde{S}' \in \widetilde{\mathcal{S}}$ intersecting a block $\widetilde{M}_k^h$ covering M_k^h, if $\widetilde{S}' \cap \widetilde{M}_k^h$ and $\widetilde{S} \cap \widetilde{M}_k^h$ are sufficiently far and do not intersect a common component of $\partial \widetilde{M}_k^h$, then a surface from $\widetilde{\mathcal{S}}$ separates $\widetilde{S}'$ from $\widetilde{S}$.
(b) For any $\widetilde{S} \in \widetilde{\mathcal{S}}$ intersecting a block $\widetilde{M}_i^g$ covering M_i^g, if $\widetilde{S}' \in \widetilde{\mathcal{S}}$ is sufficiently far from $\widetilde{S} \cap \widetilde{M}_i^g$, and $\mathrm{Stab}(\widetilde{S}') \cap \mathrm{Stab}(\widetilde{M}_i^g)$ is nontrivial, then a surface from $\widetilde{\mathcal{S}}$ separates $\widetilde{S}'$ from $\widetilde{S}$.

We consider the *dual CAT(0) cube complex* $\widetilde{X}$ associated to $\mathcal{S}$ arising from Sageev's construction. Each $\widetilde{S} \in \widetilde{\mathcal{S}}$ cuts $\widetilde{M}$ into two closed halfspaces U, V and the collection of pairs $\{U, V\}$ endows $\widetilde{M}$ with a Haglund–Paulin wallspace structure (we follow the

treatment of these ideas in [HW11, §2.1] where $U \cap V$ is allowed to be nonempty). The group $G = \pi_1 M$ acting on $\widetilde{M}$ preserves this structure and hence it acts on the associated dual CAT(0) cube complex $\widetilde{X}$. The stabiliser in G of a hyperplane in $\widetilde{X}$ coincides with a conjugate of $\pi_1 S$ for an appropriate $S \in \mathcal{S}$.

If a group G acting freely a CAT(0) cube complex $\widetilde{X}$ has a finite index subgroup G' such that $G' \backslash \widetilde{X}$ is special, then we say that the action of G on $\widetilde{X}$ is *virtually special*. We prove Theorem 1.1 using the following criterion for virtual specialness.

Criterion 2.3 ([HW10, Thm 4.1])**.** *Let G be a group acting freely on a CAT(0) cube complex $\widetilde{X}$. Suppose*

1. *there are finitely many G orbits of hyperplanes in $\widetilde{X}$,*
2. *for each hyperplane $\widetilde{A} \subset \widetilde{X}$, there are finitely many* $\mathrm{Stab}(\widetilde{A})$ *orbits of hyperplanes that intersect $\widetilde{A}$,*
3. *for each hyperplane $\widetilde{A} \subset \widetilde{X}$, there are finitely many* $\mathrm{Stab}(\widetilde{A})$ *orbits of hyperplanes that osculate with $\widetilde{A}$,*
4. *for each hyperplane $\widetilde{A} \subset \widetilde{X}$, the subgroup* $\mathrm{Stab}(\widetilde{A}) \subset G$ *is separable, and*
5. *for each pair of intersecting hyperplanes $\widetilde{A}, \widetilde{B} \subset \widetilde{X}$, the double coset* $\mathrm{Stab}(\widetilde{A})\mathrm{Stab}(\widetilde{B}) \subset G$ *is separable.*

Then the action of G on $\widetilde{X}$ is virtually special.

For each M_i^g we choose one conjugate P_i of $\pi_1 M_i^g$ in $G = \pi_1 M$. Then G is hyperbolic relative to $\{P_i\}$ (see e.g. [BW11]) and we can discuss quasiconvexity of its subgroups relative to $\{P_i\}$ (see e.g. [MPW11, Def 1.3]). Theorem 2.1(4) implies that for each $S \in \mathcal{S}$ the fundamental group $\pi_1 S$ is quasiconvex in G relative to $\{P_i\}$, by [BW11, Thm 4.16].

Let $\widetilde{M_i^g} \subset \widetilde{M}$ be the cover of M_i^g stabilised by P_i. We describe a convex P_i–invariant subcomplex $\widetilde{Y}_i \subset \widetilde{X}$ determined by $\widetilde{M_i^g}$. Let $\mathcal{U}_i$ be the family of halfspaces U in the wallspace $\widetilde{M}$ with $\mathrm{diam}(U \cap N_R(\widetilde{M_i^g})) = \infty$ for some $R > 0$, where N_R denotes the R–neighbourhood. Let $\widetilde{Y}_i \subset \widetilde{X}$ be the subcomplex spanned on the vertices whose halfspaces are in $\mathcal{U}_i$.

By [HW11, Thm 6.16 and Lem 6.17] the group G *acts cocompactly on $\widetilde{X}$ relative to* $\{\widetilde{Y}_i\}$ in the following sense: there exists a compact subcomplex $K \subset \widetilde{X}$ such that:

- $\widetilde{X} = GK \cup \bigcup_i G\widetilde{Y}_i$,
- $g\widetilde{Y}_i \cap \widetilde{Y}_j \subset GK$ unless $j = i$ and $g \in P_i$, and
- P_i acts cocompactly on $\widetilde{Y}_i \cap GK$.

Since G is hyperbolic relative to $\{P_i\}$, the intersections $g\widetilde{Y}_i \cap \widetilde{Y}_j \subset GK$ have bounded diameter. If G acts freely on $\widetilde{X}$, then GK is locally finite. It follows that each $\widetilde{Y}_i$ is *superconvex*, in the sense that there is a uniform bound on the diameter of an isometrically embedded flat strip $[-d, d] \times I$ with $[-d, d] \times \{0\} \subset \widetilde{Y}_i$ and $[-d, d] \times \{1\}$ outside $\widetilde{Y}_i$. This is proved in [HW11, Lem 6.19], where it is shown that if there is no uniform bound, then the hyperplanes dual to I limit to a hyperplane $\widetilde{A}$ stabilised by a nontrivial element of P_i. But then both halfspaces of $\widetilde{M}$ corresponding to $\widetilde{A}$ belong to $\mathcal{U}_i$. This implies $\widetilde{A} \subset \widetilde{Y}_i$ which is a contradiction.

Observe that $G = \pi_1 M$ splits as a graph of groups with transitional tori groups as edge groups. The group G is hyperbolic relative to the vertex groups $P_i = \pi_1 M_i^g$. We now explain that to prove Theorem 1.1 it suffices to complement Theorem 2.1 with the following.

Theorem 2.4 (Specialisation). *Let G be the fundamental group of a graph of groups with free-abelian edge groups. Suppose that G is hyperbolic relative to some collection of the vertex groups $\{P_i\}$. Assume that G acts cocompactly on a CAT(0) cube complex $\widetilde{X}$ relative to superconvex $\{\widetilde{Y}_i\}$. Suppose also that:*

(i) the action of G on $\widetilde{X}$ is free and satisfies finiteness conditions (1)–(3) of Criterion 2.3,
(ii) for any finite index subgroup E° of an edge group $E \subset P_i$, there is a finite index subgroup $P_i' \subset P_i$ with $P_i' \cap E \subset E^\circ$,
(iii) the action of each P_i on $\widetilde{Y}_i$ is virtually special, with finitely many orbits of codim–2–hyperplanes,
(iv) each non-parabolic vertex group is the fundamental group of a sparse cube complex that is virtually special.

Then the action of G is virtually special.

A *codim–2–hyperplane* in a CAT(0) cube complex is the intersection of a pair of intersecting hyperplanes. A cube complex is *sparse* if it is the quotient of a CAT(0) cube complex by a free action that is *cosparse*, i.e. relatively cocompact with $\widetilde{Y}_i$ *quasiflats*. A quasiflat is an associated CAT(0) cube complex dual to an action of a virtually abelian group on a wallspace with finitely many orbits of walls.

We now derive the hypothesis of Theorem 2.4 from the conclusion of Theorem 2.1. By Theorem 2.1(1), the action of $\pi_1 M$ on $\widetilde{X}$ is free. Moreover, since the family $\mathcal{S}$ is finite, Condition (1) of Criterion 2.3 is satisfied, and since $\mathcal{S}$ is in general position, we have Condition (2) of Criterion 2.3. We now deduce Condition (3) of Criterion 2.3. Hyperplanes in a CAT(0) cube complex *osculate* if they are disjoint and not separated by another hyperplane. Similarly, two disjoint surfaces $\widetilde{S}, \widetilde{S}' \in \widetilde{\mathcal{S}}$ *osculate* if there is no surface in $\widetilde{\mathcal{S}}$ separating $\widetilde{S}'$ from $\widetilde{S}$. Hence osculating hyperplanes in $\widetilde{X}$ correspond to osculating $\widetilde{S}, \widetilde{S}' \in \widetilde{\mathcal{S}}$. We need to show that there are finitely many $\mathrm{Stab}(\widetilde{S})$ orbits of surfaces in $\widetilde{\mathcal{S}}$ osculating with $\widetilde{S}$. Note that if $\widetilde{S}'$ osculates with $\widetilde{S}$, then it must intersect one of the finitely many $\mathrm{Stab}(\widetilde{S})$ orbits of graph manifold and hyperbolic blocks intersected by $\widetilde{S}$, since otherwise it would be separated from $\widetilde{S}$ by a transitional plane $\widetilde{T}$. But $\widetilde{T} \in \widetilde{\mathcal{S}}$ by Theorem 2.1(2), so $\widetilde{S}$ and $\widetilde{S}'$ would not osculate. If both $\widetilde{S}$ and $\widetilde{S}'$ intersect the same block $\widetilde{M_i^g}$, then by Strong Separation (b) of Theorem 2.1(5) they have to be at bounded distance which yields finitely many $\mathrm{Stab}(\widetilde{S})$ orbits. If $\widetilde{S}$ and $\widetilde{S}'$ do not intersect the same graph manifold block but intersect the same block $\widetilde{M_k^h}$, then by Strong Separation (a) of Theorem 2.1(5) they have to be at bounded distance which also yields finitely many $\mathrm{Stab}(\widetilde{S})$ orbits. This proves Condition (3) of Criterion 2.3. Hence Hypothesis (i) of Theorem 2.4 is satisfied.

Let $E \subset P_i$ be the stabiliser of a transitional plane $\widetilde{T} \subset \partial \widetilde{M}_i^g$ and let $E^\circ \subset E$ be a finite index subgroup. In order to verify Hypothesis (ii) of Theorem 2.4 we use the following construction.

Construction 2.5 ([PW11, Constr 4.13])**.** Let M^g be a graph manifold and let S be a surface embedded in M^g. Then there is $N_0 > 0$ such that for any multiple N of N_0 we have a finite cover $(M^g)_N^S$ with the following properties.

- The surface S lifts to $(M^g)_N^S$.
- The degree of $(M^g)_N^S \to M^g$ restricted to any JSJ or boundary torus of $(M^g)_N^S$ intersected by S is divisible by $\frac{N}{N_0}$.

By Smith normal form there is a splitting $E = \mathbb{Z} \times \mathbb{Z}$ such that $E^\circ = n_1\mathbb{Z} \times n_2\mathbb{Z}$. By Proposition 3.2 there is a finite cover $\widehat{M}_i^g$ of M_i^g and incompressible embedded surfaces $S_1, S_2 \subset \widehat{M}_i^g$ intersecting the boundary torus of $\widehat{M}_i^g$ covered by $\widetilde{T}$ in circles corresponding to the two factors above. In other words, appropriate conjugates of $\pi_1 S_1, \pi_1 S_2$ have $\pi_1 S_1 \cap E$ of finite index in $\mathbb{Z} \times \{0\}$ and $\pi_1 S_2 \cap E$ of finite index in $\{0\} \times \mathbb{Z}$. Let $(\widehat{M}_i^g)_N^{S_1}$ be the finite cover of $\widehat{M}_i^g$ from Construction 2.5 with $\frac{N}{N_0}$ divisible by n_2. The appropriate conjugate P_i° of the fundamental group of $(\widehat{M}_i^g)_N^{S_1}$ satisfies $P_i^\circ \cap E \subset \mathbb{Z} \times n_2\mathbb{Z}$. We denote by S_2° the elevation of S_2 to $(\widehat{M}_i^g)_N^{S_1}$ intersecting the quotient torus of $\widetilde{T}$. We now apply Construction 2.5 to $S_2^\circ \subset (\widehat{M}_i^g)_N^{S_1}$ and parameter divisible by n_1. The cover we obtain has fundamental group P_i' satisfying Hypothesis (ii).

The action of P_i on $\widetilde{Y}_i$ is free and obviously satisfies Condition (1) of Criterion 2.3. By the choice of $\mathcal{U}_i$ in the definition of $\widetilde{Y}_i$, any hyperplane $\widetilde{A}$ intersecting $\widetilde{Y}_i$ corresponds to a surface $\widetilde{S} \in \widetilde{\mathcal{S}}$ whose stabiliser intersects nontrivially $P_i = \mathrm{Stab}(\widetilde{M}_i^g)$. If $\widetilde{S}$ intersects $\widetilde{M}_i^g$, then Conditions (2) and (3) of Criterion 2.3 for $\widetilde{A} \cap \widetilde{Y}_i$ follow from Strong Separation (b) in Theorem 2.1(5). If $\widetilde{S}$ is disjoint from $\widetilde{M}_i^g$, we can derive Conditions (2) and (3) from Remark 3.10 and Strong Separation (a). In particular, $\widetilde{Y}_i$ has finitely many P_i orbits of codim–2–hyperplanes.

The nontrivial stabilisers in P_i of hyperplanes in $\widetilde{Y}_i$ correspond to either fundamental groups of the components of $S \cap M_i^g$, which are virtually embedded in M_i^g by Theorem 2.1(3) or subgroups of the fundamental groups of the transitional tori, which are also virtually embedded by Hypothesis (ii). Except for rank 1 subgroups of the transitional tori groups all these stabilisers are separable by [PW11, Thm 1.1] and double coset separable by [PW11, Thm 1.2]. The rank 1 case is easily handled by expressing the corresponding circles as intersections of tori and surfaces guaranteed by Proposition 3.2. Hence we have Conditions (4) and (5) of Criterion 2.3, and by Criterion 2.3 the action of P_i on $\widetilde{Y}_i$ is virtually special. This is Hypothesis (iii).

Moreover, by [Wis11, Thm 14.29] and the remark after its proof each $\pi_1 M_k^h$ is the fundamental group of a sparse virtually special cube complex, and we have Hypothesis (iv) as well.

3. Cubulation

The goal of this section is to prove Theorem 2.1 (Cubulation). To construct a family $\mathcal{S}$ in a mixed manifold satisfying Strong Separation, we will use families of surfaces in blocks satisfying *WallNbd-WallNbd Separation*:

Definition 3.1. Let $\mathcal{S}$ be a family of immersed incompressible surfaces in an aspherical compact Riemmanian manifold M. Let $\widetilde{\mathcal{S}}$ be the family of elevations of the surfaces in $\mathcal{S}$ to the universal cover $\widetilde{M}$ of M. The family $\mathcal{S}$ has *WallNbd-WallNbd Separation* if for any $R > 0$ there is $W_R > 0$ such that if $\widetilde{S}, \widetilde{S}' \in \widetilde{\mathcal{S}}$ have neighbourhoods $N_R(\widetilde{S}), N_R(\widetilde{S}')$ at distance $\geq W_R$, then these neighbourhoods are separated by a surface in $\widetilde{\mathcal{S}}$. This property does not depend on the choice of the Riemannian metric, but the value of W_R might vary.

3.1. **Graph manifold blocks.** Let M^g be a *graph manifold*, i.e. a compact connected oriented irreducible 3–manifold with only Seifert fibered blocks in its JSJ decomposition. Assume additionally $\partial M^g \neq \emptyset$. If M^g is Seifert fibered, then an immersed incompressible surface $S_o \to M^g$ is *horizontal* if it is transverse to the fibers and *vertical* if it is a union of fibers. An immersed incompressible surface $S \to M^g$ is assumed to be homotoped so that its pieces are horizontal or vertical. Every graph manifold has a finite cover whose underlying graph is *simple*, i.e. has no double edges or edges connecting a vertex to itself. Moreover, one can additionally require that the finite cover is *simple*, i.e. either $T \times I$ or else having the property that every block is a product of the fiber with a base surface of genus ≥ 1. We first review the existence result for surfaces in graph manifolds with boundary from [PW11].

Proposition 3.2. *Let M^g be a graph manifold with nonempty boundary. There exists a finite cover $\widehat{M}^g$ of M^g such that for each circle C in a boundary or JSJ torus $T \subset \widehat{M}^g$ there is an incompressible surface S_C embedded in $\widehat{M}^g$ with $S_C \cap T$ consisting of a nonempty set of circles parallel to C.*

We construct $\widehat{M}^g$ in the proof of [PW11, Prop 3.1]: assuming that M^g is simple and its underlying graph is simple, the cover $\widehat{M}^g \to M^g$ is of degree 2^k, where k is the number of blocks of M^g.

Consider a family $\mathcal{S}^g$ of surfaces in $\widehat{M}^g$ obtained by collecting surfaces satisfying Proposition 3.2 for each vertical C, and adding vertical tori in each block $\widehat{B}$ so that their base curves *fill* the base surface $\widehat{\Sigma}$. To *fill* means that for every non-vertical element of $\pi_1\widehat{\Sigma}$ one of these tori is a cut-surface. In the exceptional case where M^g is Seifert fibered we add a horizontal surface to $\mathcal{S}^g$. Then $\mathcal{S}^g$ satisfies the following:

Corollary 3.3 ([PW11, Prop 3.1])**.** *Let M^g be a graph manifold with nonempty boundary. There exists a finite cover $\widehat{M}^g$ with a finite family $\widehat{\mathcal{S}}^g$ of embedded incompressible surfaces such that its projection $\mathcal{S}^g$ consisting of immersed incompressible surfaces in M^g satisfies the following. For each element of $\pi_1 B$ of a block B of M^g, a component of $S^g \cap B$ is a cut-surface for some $S^g \in \mathcal{S}^g$.*

In particular, if M^g is simple, then for each block B of M^g with base Σ the fundamental group $\pi_1\Sigma$ acts freely on the dual CAT(0) cube complex associated to the vertical pieces of the restriction of $\mathcal{S}^g$ to B.

Remark 3.4. As either Σ is an annulus or $\chi(\Sigma) < 0$, the associated action of $\pi_1\Sigma$ is cocompact. Hence the family of vertical pieces of the restriction of $\mathcal{S}^g$ to B satisfies WallNbd-WallNbd Separation in B. This is a variant of Lemma 3.8.

By the properties of arcs on hyperbolic surfaces we also have the following.

Remark 3.5. There exists $D > 0$ with the following property. Let $\widetilde{S}, \widetilde{S}'$ be elevations to $\widetilde{M}^g$ of surfaces in $\mathcal{S}^g$. Suppose that there is a block $\widetilde{B} \subset \widetilde{M}^g$ such that $\widetilde{S}_o = \widetilde{S} \cap \widetilde{B}$ and $\widetilde{S}'_o = \widetilde{S}' \cap \widetilde{B}$ are both vertical. Assume also that there is a plane $\widetilde{T} \subset \partial\widetilde{B}$ intersecting both $\widetilde{S}_o$ and $\widetilde{S}'_o$. If the distance between the lines $\widetilde{S}_o \cap \widetilde{T}$ and $\widetilde{S}'_o \cap \widetilde{T}$ is $\geq D$ in the intrinsic metric of $\widetilde{T}$, then $\widetilde{S}_o$ and $\widetilde{S}'_o$ are disjoint and $\mathcal{T}(\widetilde{S}_o) \cap \mathcal{T}(\widetilde{S}'_o) = \widetilde{T}$.

Finally, we will need the following.

Lemma 3.6. *Let S be an incompressible surface embedded in a graph manifold M^g. Let $S' \to S$ be a finite cover. Then $S' \to M^g$ is virtually embedded.*

Proof. This follows immediately from [RW98, Thm 2.3]. □

3.2. **Hyperbolic blocks.** We now review the existence result for surfaces in hyperbolic blocks. First we invoke a hyperbolic analogue of Corollary 3.3.

Theorem 3.7 ([Wis11, Cor 14.33])**.** *Let M^h be a compact hyperbolic 3–manifold with nonempty boundary. There is in M^h a finite family $\mathcal{S}^h$ of geometrically finite immersed incompressible surfaces such that for each element of $\pi_1 M^h$ there is a cut-surface in $\mathcal{S}^h$.*

Consequently, $\pi_1 M^h$ acts freely on the associated dual CAT(0) cube complex. If we extend the above family $\mathcal{S}^h$ to any larger finite family of geometrically finite immersed incompressible surfaces, the action on the associated dual CAT(0) cube complex is free as before. It is also relatively cocompact by [HW11, Thm 6.16]. It is cosparse since the parabolic subgroups are free-abelian. The conclusion of the following lemma strengthens its cocompact variant mentioned in Remark 3.4.

Lemma 3.8. *Let $\mathcal{S}$ be a family of immersed incompressible surfaces in a compact aspherical 3–manifold M. Suppose that the action of $\pi_1 M$ on the associated dual CAT(0) cube complex is free and cosparse. Then $\mathcal{S}$ has WallNbd-WallNbd Separation.*

Proof. Let $\widetilde{\mathcal{S}}$ be the family of elevations to the universal cover $\widetilde{M}$ of M of the surfaces in $\mathcal{S}$. Denote by $\widetilde{X}$ the associated dual CAT(0) cube complex. Since the action is cosparse, there is a $\pi_1 M$–invariant cocompact subcomplex $\widetilde{X}_c \subset \widetilde{X}$ which is isometrically embedded w.r.t. to the 1–skeleton metric and with the property that if a pair of hyperplanes intersects in $\widetilde{X}$, then their restrictions to $\widetilde{X}_c$ intersect as well. (This *isometric core property* follows from [Wis11, Lem 16.6], where we take $\widetilde{K}$ to intersect all codim–2–hyperplanes.) Hence osculating hyperplanes in $\widetilde{X}_c$ correspond to osculating surfaces in

$\widetilde{\mathcal{S}}$. Thus for every R there is D_R such that if two hyperplanes in $\widetilde{X}_c$ are at distance $\geq D_R$, then their corresponding elevations of surfaces in $\widetilde{\mathcal{S}}$ are at distance $\geq R$.

Since the action of $\pi_1 M$ on $\widetilde{X}_c$ is free and cocompact, we also have the converse: for every D_R there is W_R such that if two surfaces in $\widetilde{\mathcal{S}}$ are at distance $\geq W_R$, then their corresponding hyperplanes in $\widetilde{X}_c$ are at distance $\geq 2D_R$. Then the constant W_R is the required WallNbd-WallNbd Separation constant: for a given pair of surfaces $\widetilde{S}, \widetilde{S}' \in \widetilde{\mathcal{S}}$ at distance $\geq W_R$ the hyperplane at distance $\geq D_R$ separating the two hyperplanes in $\widetilde{X}_c$ corresponding to $\widetilde{S}$ and $\widetilde{S}'$ corresponds to a surface separating $N_R(\widetilde{S})$ from $N_R(\widetilde{S}')$. $\square$

We have also the following analogue of Remark 3.5.

Lemma 3.9. *Let $\mathcal{S}^h$ be a finite family of geometrically finite immersed incompressible surfaces in a compact hyperbolic 3–manifold M^h. There exists $D > 0$ such that if two elevations $\widetilde{S}, \widetilde{S}'$ to $\widetilde{M}^h$ of surfaces in $\mathcal{S}^h$ intersect a boundary plane $\widetilde{T}$ in lines at distance $\geq D$ in the intrinsic metric of $\widetilde{T}$, then $\widetilde{S}$ and $\widetilde{S}'$ are disjoint and $\mathcal{T}(\widetilde{S}) \cap \mathcal{T}(\widetilde{S}') = \widetilde{T}$.*

Proof. We can assume that the Riemannian metric on M^h is hyperbolic and the toroidal boundary components are horospherical. All elevations $\widetilde{S}$ of surfaces in $\mathcal{S}^h$ are L–quasiconvex in $\widetilde{M}^h$ with L depending only on $\mathcal{S}^h$. There is a constant $D(L)$ such that if $\widetilde{S}$ intersects $\widetilde{T}$, then nearest point projection of $\widetilde{M}^h$ onto $\widetilde{T}$ maps $\widetilde{S} \cup (\mathcal{T}(\widetilde{S}) - \widetilde{T})$ into the $\frac{1}{2}D(L)$–neighbourhood of the line $\widetilde{T} \cap \widetilde{S}$. We can then take $D = D(L)$. $\square$

The following easy variation of Lemma 3.9 also holds.

Remark 3.10. In the setting of Lemma 3.9, suppose the surface $\widetilde{S}'$ does not intersect $\widetilde{T}$, but their stabilisers have nontrivial intersection. In that case, if $\widetilde{S}'$ is sufficiently far from $\widetilde{S} \cap \widetilde{T}$, then $\widetilde{S}$ and $\widetilde{S}'$ are disjoint and $\mathcal{T}(\widetilde{S})$ and $\mathcal{T}(\widetilde{S}')$ are disjoint.

We will need one more crucial piece of information concerning the existence of surfaces in hyperbolic blocks with designated boundary circles. We denote by $\partial_t M^h \subset \partial M^h$ the union of toroidal boundary components.

Proposition 3.11. *Let M^h be a compact hyperbolic 3–manifold and let C be a circle in a boundary torus of M^h. There exists a geometrically finite immersed incompressible surface S in M^h such that $S \cap \partial_t M^h$ is nonempty and covers C.*

In the proof of Proposition 3.11 we will use the following.

Lemma 3.12 ([Wis11, 17.12])**.** *Let G be the fundamental group of a sparse virtually special cube complex. Suppose that G is hyperbolic relative to free-abelian subgroups $\{P_i\}$ stabilising the quasiflats. Let H be a corank 1 subgroup of P_0. There exists a finite index subgroup $G' \subset G$ and a homomorphism $f : G' \to \mathbb{Z}$ such that*

- *$f(gP_ig^{-1} \cap G')$ is trivial for each $g \in G$ and $i \geq 1$,*
- *$f(gHg^{-1} \cap G')$ is trivial for each $g \in G$, and*
- *$f(P_0 \cap G')$ is nontrivial.*

Consequently, we have the following.

Corollary 3.13. *Let M^h be a compact hyperbolic 3–manifold and let C be a circle in a torus $T \subset \partial_t M^h$. There exists a finite cover $\widehat{M}^h$ of M^h and a map $f\colon \widehat{M}^h \to S^1$ such that*

- *f is homotopically trivial on elevations of boundary tori distinct from T,*
- *f is homotopically trivial on elevations of C, and*
- *f is homotopically nontrivial on a based elevation of T.*

Proof of Proposition 3.11. By passing to a finite cover we can assume that M^h has at least two boundary tori. Consider the finite cover $\widehat{M}^h$ of M^h and the map $f\colon \widehat{M}^h \to S^1$ guaranteed by Corollary 3.13. After homotoping f, we can assume that if f is homotopically trivial on an elevation of a boundary torus, then f maps it to a point. By Sard's theorem, there is a point $x \in S^1$ so that $\widehat{S} = f^{-1}(x)$ is a surface. The surface $\widehat{S}$ is disjoint from all elevations of boundary tori on which f is homotopically trivial. Hence each component C' of $\widehat{S} \cap \partial_t \widehat{M}^h$ is contained in a boundary torus $\widehat{T}$ covering T. Note that f is homotopically nontrivial on $\widehat{T}$, while homotopically trivial on C' as well as on the elevations of C to $\widehat{T}$. It follows that C' covers C. Obviously $\widehat{S}$ is oriented since it inherits a normal direction from S^1. After possibly surgering $\widehat{S}$ with compressing discs, we obtain an incompressible surface S with the same boundary. The surface S is not a virtual fiber since it misses a boundary torus in $\widehat{M}^h$. Hence S is geometrically finite. Projecting S to M^h gives the required immersed incompressible surface. $\square$

3.3. **Merging surfaces.** In this subsection we combine the surfaces described in the graph manifold blocks and hyperbolic blocks. To prove Theorem 2.1 we need the following:

Lemma 3.14. *Let S be a connected compact surface with $\chi(S) < 0$. There exists $K = K(S)$ such that for each assignment of a positive integer n_C to each boundary circle C in ∂S, there is a connected finite cover $\widehat{S}$ whose degree on each component of the preimage of C equals Kn_C.*

We can allow S to be disconnected. We can also allow annular components, but obviously required that the integers n_C coincide for both boundary circles of such a component.

Proof. Let $K = K(S)$ be the degree of a cover of S with nonzero genus. The lemma follows from [PW11, Lem 4.7]. $\square$

Proof of Theorem 2.1. The proof has two steps. In the first step we construct a family $\mathcal{S}$ of surfaces which satisfies Parts (1)–(4) of Theorem 2.1. In the second step we prove that $\mathcal{S}$ satisfies the Strong Separation property in Part (5).

Construction. Let $\mathcal{S}^h_k$ be the family of surfaces in M^h_k given by Theorem 3.7. Let $\mathcal{C}$ be the family of circles embedded in the transitional tori of M that are covered by the boundary circles of the surfaces in $\{\mathcal{S}^h_k\}$ up to homotopy on the tori. Let $\mathcal{C}_i \subset \mathcal{C}$ be the circles lying in ∂M^g_i.

By Proposition 3.2 each circle $C \in \mathcal{C}_i$ is covered by a boundary circle of an immersed incompressible surface $S^g_C \to M^g_i$ virtually embedded in M^g_i. Let $\mathcal{S}^g_i$ be the family of surfaces in M^g_i provided by Corollary 3.3 and let $\mathcal{S}'^g_i = \mathcal{S}^g_i \cup \{S^g_C\}_{C\in\mathcal{C}_i}$.

Let $\mathcal{C}'$ denote the family of circles embedded in the transitional tori of M covered (up to homotopy) by the boundary circles of the surfaces in $\{\mathcal{S}_i'^g\}$. Let $\mathcal{C}_k' \subset \mathcal{C}'$ be the circles lying in ∂M_k^h. By Proposition 3.11, for each circle $C \in \mathcal{C}_k'$ there is a geometrically finite immersed incompressible surface $S_C^h \to M_k^h$ such that $S_C^h \cap \partial_t M_k^h$ is nonempty and covers C. Let $\mathcal{S}_k'^h = \mathcal{S}_k^h \cup \{S_C^h\}_{C\in\mathcal{C}_k'}$.

We will apply Lemma 3.14 to produce families of surfaces $\{\widehat{\mathcal{S}}_k'^h\}, \{\widehat{\mathcal{S}}_i'^g\}$ covering $\{\mathcal{S}_k'^h\}, \{\mathcal{S}_i'^g\}$ such that $\mathcal{S}_o = \{\widehat{\mathcal{S}}_k'^h\} \cup \{\widehat{\mathcal{S}}_i'^g\}$ has the following property: There is a uniform $d > 0$ such that for each circle in $\mathcal{C}'$, each component of its preimage in a surface in $\mathcal{S}_o$ covers it with degree d. In order to arrange this, for a boundary circle C of a surface in $\{\mathcal{S}_k'^h\} \cup \{\mathcal{S}_i'^g\}$ let d_C denote the degree with which C maps onto a circle in $\mathcal{C}'$. Let $n_C = \frac{1}{d_C}\prod_C d_C$. Applying Lemma 3.14 with this choice of $\{n_C\}$ provides the uniform $d = K\prod_C d_C$. Note that for an annular surface the degrees d_C coincide and hence the numbers n_C coincide. We can then take a cyclic cover.

We will now extend each surface $S_o \in \mathcal{S}_o$ to a surface immersed properly in M by combining appropriately many copies of other surfaces in $\mathcal{S}_o$. First assume $S_o \in \widehat{\mathcal{S}}_i'^g$. Let $\mathcal{C}_o' \subset \mathcal{C}'$ denote the set of circles covered by the boundary components of S_o and let $m_{C'}$ denote the number of components mapping to the circle $C' \in \mathcal{C}_o'$. Denote by $\widehat{S}_{C'}^h$ the surface in $\mathcal{S}_o$ covering $S_{C'}^h$ and by $l_{C'}$ the number of boundary components of $\widehat{S}_{C'}^h$. Let $N = \prod_{C'\in\mathcal{C}_o'} l_{C'}$. Take $2N$ copies of S_o and $2m_{C'}\frac{N}{l_{C'}}$ copies of $\widehat{S}_{C'}^h$, for each C', with two opposite orientations. These surfaces combine to form a desired immersed incompressible surface extending S_o. Note that for each $C' \in \mathcal{C}'$ the surface $\widehat{S}_{C'}^h$ appears within such extension of some surface $S_o \in \widehat{\mathcal{S}}_i'^g$.

Hence it remains to consider the case $S_o \in \widehat{\mathcal{S}}_k^h \subset \widehat{\mathcal{S}}_k'^h$, where $\widehat{\mathcal{S}}_k^h$ is the family of surfaces covering the surfaces in $\mathcal{S}_k^h$. This case is treated similarly to the previous one. Let $\mathcal{C}_o \subset \mathcal{C}$ be the set of circles covered by the boundary components of S_o. Consider all the surfaces $\widehat{S}_C^g$ covering S_C^g for $C \in \mathcal{C}_o$. Let $\mathcal{C}_o' \subset \mathcal{C}'$ denote the set of circles covered by the boundary components of these surfaces $\widehat{S}_C^g$. Consider all the surfaces $\widehat{S}_C^h$, where $C \in \mathcal{C}_o' - \mathcal{C}_o$. Gluing appropriate number of copies of $S_o, \widehat{S}_C^g$, and $\widehat{S}_C^h$ gives the desired extension.

We denote the union of both of these families of extended surfaces together with the family of the JSJ tori by $\mathcal{S}$. So $\mathcal{S}$ obviously satisfies Theorem 2.1(2). Observe that Theorem 2.1(1) follows from Theorem 2.1(2) and the existence of cut-surfaces in Theorem 3.7 and Corollary 3.3. The surfaces in $\widehat{\mathcal{S}}_i'^g$ are virtually embedded in M_i^g by Lemma 3.6, hence $\mathcal{S}$ satisfies Theorem 2.1(3). The surfaces in $\widehat{\mathcal{S}}_k'^h$ are geometrically finite and thus $\mathcal{S}$ satisfies Theorem 2.1(4).

Strong Separation. We now verify Theorem 2.1(5). Let D be a constant satisfying Lemma 3.9 and Remark 3.5 for all hyperbolic and Seifert fibered blocks of M with respect to the family of surfaces used to build $\mathcal{S}$. We first prove Strong Separation (b). Let $\widetilde{S} \in \widetilde{\mathcal{S}}$ and let $\widetilde{M_i^g} \subset \widetilde{M}$ be a graph manifold block intersected by $\widetilde{S}$. We need to show that a surface $\widetilde{S}' \in \widetilde{\mathcal{S}}$ with nontrivial $\mathrm{Stab}(\widetilde{S}') \cap \mathrm{Stab}(\widetilde{M_i^g})$ and sufficiently far from $\widetilde{S} \cap \widetilde{M_i^g}$ is separated from $\widetilde{S}$ by another surface in $\widetilde{\mathcal{S}}$.

First consider the case where $\widetilde{S}'$ intersects a JSJ or boundary plane $\widetilde{T}$ of $\widetilde{M_i^g}$ intersected by $\widetilde{S}$. Since by Theorem 2.1(1) the components of $S \cap M_i^g$ are virtually embedded, there is $h \in \mathrm{Stab}(\widetilde{T})$ such that the surfaces $\widetilde{S} \cap \widetilde{M_i^g}$ and $h\widetilde{S} \cap \widetilde{M_i^g}$ are disjoint. Moreover, by passing to a power of h we can assume that they are at distance $\geq D$. Let $\widetilde{M}_{\mathrm{hor}} \subset \widetilde{M_i^g}$ be the maximal graph manifold containing $\widetilde{T}$ such that $\widetilde{S}$ is horizontal in all the blocks of $\widetilde{M}_{\mathrm{hor}}$. In the extreme cases $\widetilde{M}_{\mathrm{hor}}$ can equal $\widetilde{M_i^g}$ or $\widetilde{T}$. Let $\widetilde{N}$ be the union of $\widetilde{M}_{\mathrm{hor}}$ with the adjacent hyperbolic and Seifert fibered blocks. By Lemmas 3.9 and 3.5 the surfaces $\widetilde{S} \cap \widetilde{N}$ and $h\widetilde{S} \cap \widetilde{N}$ are disjoint and the boundary lines of $\widetilde{S} \cap \widetilde{N}$ and $h\widetilde{S} \cap \widetilde{N}$ do not intersect a common JSJ plane. Hence the entire $\widetilde{S}$ and $h\widetilde{S}$ are disjoint.

For fixed h, depending on $\widetilde{S}$ and $\widetilde{T}$, the surface $h\widetilde{S} \cap \widetilde{M}_{\mathrm{hor}}$ is in a bounded neighbourhood of $\widetilde{S} \cap \widetilde{M}_{\mathrm{hor}}$. Hence if $\widetilde{S}' \cap \widetilde{M_i^g}$ is sufficiently far from $\widetilde{S} \cap \widetilde{M_i^g}$, then $\widetilde{S}' \cap \widetilde{M}_{\mathrm{hor}}$ is at distance $\geq D$ from each of $h^{\pm 1}\widetilde{S} \cap \widetilde{M}_{\mathrm{hor}}$. As before $\widetilde{S}'$ is disjoint from both $h^{\pm 1}\widetilde{S}$, and one of $h^{\pm 1}\widetilde{S}$ separates $\widetilde{S}'$ from $\widetilde{S}$, as desired. Since there are finitely many $\mathrm{Stab}(\widetilde{S} \cap \widetilde{M_i^g})$ orbits of JSJ or boundary planes $\widetilde{T}$ of $\widetilde{M_i^g}$ intersected by $\widetilde{S}$, this argument works for all $\widetilde{T}$ simultaneously.

Secondly, we consider the similar case where $\widetilde{S}'$ is disjoint from $\widetilde{M_i^g}$ but for a JSJ plane $\widetilde{T}$ in $\partial\widetilde{M_i^g}$ the intersection $\mathrm{Stab}(\widetilde{S}') \cap \mathrm{Stab}(\widetilde{T})$ is nontrivial. If $\widetilde{S}'$ is sufficiently far from $\widetilde{S} \cap \widetilde{M_i^g}$ and hence far from $h^{\pm 1}\widetilde{S} \cap \widetilde{T}$, then by Remark 3.10 one of the surfaces $h^{\pm 1}\widetilde{S}$ separates $\widetilde{S}'$ from $\widetilde{S}$.

To complete the proof of Strong Separation (b) it remains to consider a third case where $\widetilde{S}'$ intersects a Seifert fibered block $\widetilde{M}_o \subset \widetilde{M_i^g}$ intersected by $\widetilde{S}$, but is disjoint from the JSJ and boundary planes of $\widetilde{M_i^g}$ intersected by $\widetilde{S}$. In that case the pieces of $\widetilde{S}$ and $\widetilde{S}'$ in $\widetilde{M}_o$ are vertical. Since the proof for Strong Separation (a) is the same, we perform it simultaneously: in that case $\widetilde{M}_o$ denotes the hyperbolic block $\widetilde{M_k^h}$. In both cases if we denote by $\mathcal{T}(\widetilde{S} \cap \widetilde{M}_o)$ the set of JSJ and boundary planes in $\partial\widetilde{M}_o$ intersecting $\widetilde{S}$, then $\mathcal{T}(\widetilde{S} \cap \widetilde{M}_o)$ and $\mathcal{T}(\widetilde{S}' \cap \widetilde{M}_o)$ are disjoint.

As before, for any JSJ or boundary plane $\widetilde{T} \in \mathcal{T}(\widetilde{S} \cap \widetilde{M}_o)$ there is $h \in \mathrm{Stab}(\widetilde{T})$ such that $\widetilde{S}$ and $h\widetilde{S}$ are disjoint. The same holds with $\widetilde{S}$ replaced by $\widetilde{S}'$. There is $D' > 0$ such that for each $\widetilde{T}$, the translate $h\widetilde{S} \cap \widetilde{T}$ is contained in the D'–neighbourhood of $\widetilde{S} \cap \widetilde{T}$ in the intrinsic metric of $\widetilde{T}$, and the same property holds with $\widetilde{S}$ replaced by $\widetilde{S}'$. Let W_R be a WallNbd-WallNbd Separation constant guaranteed by Lemma 3.8 and Remark 3.4 for $R = D' + D$ in all hyperbolic and Seifert fibered blocks with respect to the family of surfaces used to build $\mathcal{S}$.

If the piece $\widetilde{S}' \cap \widetilde{M}_o$ is at distance $\geq 2R + W_R$ from the piece $\widetilde{S} \cap \widetilde{M}_o$, then by WallNbd-WallNbd Separation there is a surface $\widetilde{S}^* \in \widetilde{\mathcal{S}}$ such that $\widetilde{S}^* \cap \widetilde{M}_o$ separates $N_R(\widetilde{S}' \cap \widetilde{M}_o)$ from $N_R(\widetilde{S} \cap \widetilde{M}_o)$ in $\widetilde{M}_o$. If $\mathcal{T}(\widetilde{S}^* \cap \widetilde{M}_o)$ is disjoint from $\mathcal{T}(\widetilde{S}' \cap \widetilde{M}_o) \cup \mathcal{T}(\widetilde{S} \cap \widetilde{M}_o)$, then $\widetilde{S}^*$ is disjoint from $\widetilde{S}'$ and $\widetilde{S}$ and separates them, as desired.

Otherwise, if $\mathcal{T}(\widetilde{S}^* \cap \widetilde{M}_o)$ intersects $\mathcal{T}(\widetilde{S}' \cap \widetilde{M}_o) \cup \mathcal{T}(\widetilde{S} \cap \widetilde{M}_o)$, we can assume without loss of generality that there is a JSJ or boundary plane $\widetilde{T} \in \mathcal{T}(\widetilde{S}^* \cap \widetilde{M}_o) \cap \mathcal{T}(\widetilde{S}' \cap \widetilde{M}_o)$. By the definition of R and D', there is a translate $h\widetilde{S}'$ disjoint from $\widetilde{S}'$ such that $h\widetilde{S}' \cap \widetilde{T}$

separates $\widetilde{S}' \cap \widetilde{T}$ from $N_D(\widetilde{S}^* \cap \widetilde{T})$ in the intrinsic metric. Moreover, by Lemma 3.9 or Remark 3.5 the surface $h\widetilde{S}' \cap \widetilde{M}_o$ is disjoint from $\widetilde{S}^* \cap \widetilde{M}_o$ and $\mathcal{T}(h\widetilde{S}' \cap \widetilde{M}_o)$ intersects $\mathcal{T}(\widetilde{S}^* \cap \widetilde{M}_o)$ only in $\widetilde{T}$. Hence $h\widetilde{S}'$ and $\widetilde{S}$ are disjoint and $h\widetilde{S}'$ separates $\widetilde{S}'$ from $\widetilde{S}$, as desired. □

4. Separability in special cube complexes

4.1. Special cube complexes.

Definition 4.1 (compare [HW08, Def 3.2]). Let X be a nonpositively curved cube complex. A *midcube* of an n–cube $[-1, 1]^n = I^n$ is the subspace obtained by restricting exactly one coordinate to 0. Let $\mathcal{M}$ denote the disjoint union of all the midcubes of X. An *immersed hyperplane* of X is a connected component of the quotient of $\mathcal{M}$ by the inclusion maps.

An immersed hyperplane A of X *self-intersects* if it contains two different midcubes of the same cube of X. If A does not self-intersect, then it embeds into X, and is called a *hyperplane*.

Similarly one defines *immersed codim–2–hyperplanes*, which in a CAT(0) cube complex are intersections of pairs of intersecting hyperplanes.

An edge e is *dual* to an immersed hyperplane A if A contains the midcube of e. A hyperplane A is *two-sided* if one can orient all of its dual edges so that any two that are parallel in a square s of X are oriented consistently within s.

If a hyperplane A is two-sided and we orient its dual edges as above, we say that A *directly self-osculates*, if it has two dual edges with the same initial vertex or with the same terminal vertex. If A is two-sided and the initial vertex of one of its dual edges coincides with the terminal vertex of another or the same dual edge, then A *self-osculates indirectly*.

Distinct hyperplanes A, B *interosculate*, if there are dual edges e_1, e_2 of A and f_1, f_2 of B such that e_1, f_1 lie in a square and e_2, f_2 share a vertex but do not lie in a square.

A nonpositively curved cube complex is *special*, if its immersed hyperplanes do not self-intersect, are two-sided, do not directly self-osculate or inter-osculate.

A group is *special* if it is a fundamental group of a special cube complex with finitely many hyperplanes.

Note that we do not require special cube complexes to be compact. However, in this article we will always assume that they have finitely many hyperplanes.

Theorem 4.2 ([HW08, Thm 4.2]). *A special cube complex X with finitely many hyperplanes admits a local isometry $X \to R(X)$ into the Salvetti complex $R(X)$ of a finitely generated right-angled Artin.*

The generators of the Artin group correspond to the hyperplanes of X. Each edge of X dual to a hyperplane $A \subset X$ is mapped by the local isometry to an edge of $R(X)$ labeled by the generator corresponding to A. Note that $R(X)$ is compact special.

Our goal is to revisit and strengthen hyperplane separability and double hyperplane separability established in [HW08] for compact special cube complexes. The starting point and the main tool is the following.

Theorem 4.3 ([HW08, Cor 6.7])**.** *Let $Y \to X$ be a local isometry of a compact cube complex Y into a special cube complex X. There is a finite cover $\widehat{X} \to X$, called the* canonical completion *of $Y \to X$, to which Y lifts and a* canonical retraction *map $\widehat{X} \to Y \subset \widehat{X}$, restricting to identity on Y, which is possibly not cellular, but continuous and maps hyperplanes of $\widehat{X}$ intersecting Y into themselves.*

If one first subdivides X (or takes an appropriate cover) to eliminate indirect self-osculations then the retraction can be made cellular.

All paths we discuss in X are assumed to be combinatorial. Let $\|X\|$ denote the minimum of the lengths of essential closed paths in X.

Lemma 4.4. *Let X be a special cube complex with finitely many hyperplanes. Then for each $d > 0$ there is a finite cover $\widehat{X}$ of X with $\|\widehat{X}\| > d$.*

Note that this property is preserved under passing to further covers.

Proof of Lemma 4.4. Let $X \to R = R(X)$ be a local isometry into the Salvetti complex of a finitely generated right-angled Artin group F coming from Theorem 4.2. Since R is compact, there is a finite set $\mathcal{F}$ of conjugacy classes of elements of F which can be represented by closed paths of length $\leq d$ in R. Since F is residually finite, it has a finite index subgroup $\widehat{F}$ disjoint from the set of elements whose classes lie in $\mathcal{F}$. Let $\widehat{R} \to R$ be the finite cover corresponding to $\widehat{F} \subset F$. Let $\widehat{X} \to X$ be its pullback. Since $\widehat{X} \to \widehat{R}$ is a local isometry, it is π_1–injective and we have $\|\widehat{X}\| > d$ as desired. □

4.2. **Separability.** Recall that a subgroup H of a group G is *separable* if for each $g \in G - H$, there is a finite index subgroup F of G with $g \notin FH$.

Definition 4.5. Let X be a nonpositively curved cube complex and $\widetilde{X}$ its universal cover. Let A be an immersed hyperplane in X with an elevation $\widetilde{A}$ in $\widetilde{X}$. The *carrier* $N(\widetilde{A})$ is the smallest subcomplex of $\widetilde{X}$ containing $\widetilde{A}$. It is isomorphic with $\widetilde{A} \times I$. The *carrier* $N(A)$ is the quotient of $N(\widetilde{A})$ by $\mathrm{Stab}(\widetilde{A})$. Then $N(A)$ maps to X. If A is two-sided and does not self-osculate (directly or indirectly), then $N(A)$ embeds in X and we identify it with its image. We similarly define carriers of immersed codim–2–hyperplanes.

A path $\alpha \to X$ *starting (resp. ending) at a vertex v of $N(A)$* is a path that starts (resp. ends) at the image of v in X. The path α is *in* $N(A)$ if it lifts to a path in $N(A)$. The path α is path-homotopic *into* $N(A)$ if it is path-homotopic to a path in $N(A)$.

Definition 4.6. An immersed hyperplane A in a cube complex X has *injectivity radius* $> d$ if all paths of length $\leq 2d$ in X starting and ending at $N(A)$ are path-homotopic into $N(A)$. In particular if $d > 0$, then A does not self-intersect or osculate. Equivalently, all elevations of $N(A)$ to the universal cover $\widetilde{X}$ of X are at distance $> 2d$.

Lemma 4.7. *Let X be a special cube complex and let $A \subset X$ be one of its finitely many hyperplanes. Then for each d there is a finite cover $\widehat{X} \to X$ such that any elevation $\widehat{A} \subset \widehat{X}$ of A has injectivity radius $> d$.*

In the compact case, Lemma 4.7 and the following consequence, was proved in [HW08, Cor 9.7], using Theorem 4.3. Note that the conclusion of Lemma 4.7 is preserved under passing to further covers.

Corollary 4.8. *Let G be the fundamental group of a virtually special cube complex with finitely many hyperplanes and let $H \subset G$ be the fundamental group of an immersed hyperplane. Then H is separable in G.*

Proof of Lemma 4.7. Like before, let $X \to R = R(X)$ be the local isometry into the Salvetti complex of the finitely generated right-angled Artin group F coming from Theorem 4.2. Let T be the image in R of the hyperplane A. Since R is compact, it admits finitely many paths starting and ending at $N(T)$ of length $\leq 2d$, not path-homotopic into $N(T)$. Let $\mathcal{F}$ denote the family of conjugacy classes determined by closing them up by paths in $N(T)$. Then $\mathcal{F}$ is a union of classes determined by finitely many nontrivial cosets of the form Hg, where $H = \pi_1 T$. Since hyperplane subgroups in F are separable [HW08, Cor 9.7], there is a finite index subgroup $\widehat{F} \subset F$ disjoint from the set of elements whose classes lie in $\mathcal{F}$. Again let $\widehat{R} \to R$ be the finite cover corresponding to $\widehat{F} \subset F$ and let $\widehat{X} \to X$ be its pullback.

We verify that $\widehat{X}$ is the desired cover. The universal cover $\widetilde{X}$ of X embeds into the universal cover $\widetilde{R}$ of R as a convex subcomplex. Since $\pi_1\widehat{X} \subset \pi_1\widehat{R}$, the $\pi_1\widehat{X}$ orbit of a hyperplane $\widetilde{A} \subset \widetilde{X}$ is contained in the $\pi_1\widehat{R}$ orbit of a hyperplane $\widetilde{T} \subset \widetilde{R}$. Since $\pi_1\widehat{R}$ translates of hyperplane carriers in $\widetilde{R}$ are at distance $> 2d$, so are the $\pi_1\widehat{X}$ translates of hyperplane carriers in $\widehat{X}$. □

4.3. **Separability of double cosets.** We now turn to discussing "double coset separability". Let $H_1, H_2 \subset G$ be two subgroups of a group G. The *double coset* H_1H_2 is *separable*, if for each $g \in G - H_1H_2$ there is a finite index subgroup F of G with $g \notin FH_1H_2$.

Definition 4.9. Let A be a hyperplane in a nonpositively curved cube complex X. Let $\widetilde{A}$ be an elevation of A to the universal cover $\widetilde{X}$ of X. Let $\widetilde{A}^{+d} \subset \widetilde{X}$ be the combinatorial ball of radius d around the carrier of $\widetilde{A}$. We say that A is *d–locally finite*, if $\widetilde{A}^{+d}$ has finitely many Stab(A) orbits of hyperplanes.

In particular, A is 0–locally finite if there are finitely many Stab(A) orbits of hyperplanes intersecting $\widetilde{A}$. If additionally there are finitely many Stab(A) orbits of hyperplanes osculating with $\widetilde{A}$, then A is 1–locally finite.

Lemma 4.10. *Let G be the fundamental group of a special cube complex with finitely many hyperplanes. Let $H_1, H_2 \subset G$ be conjugates of the fundamental groups of hyperplanes one of which is d–locally finite for all d. Then the double coset H_1H_2 is separable in G.*

While Lemma 4.10 could be avoided here, we include it to shed more light on double hyperplane separability.

Proof. Let $\widetilde{X}$ be the universal cover of the special cube complex X from the lemma. Let $\widetilde{A}, \widetilde{B} \subset \widetilde{X}$ be the hyperplanes stabilised by H_1, H_2. Let $A, B \subset X$ be the projections of $\widetilde{A}, \widetilde{B}$. Without loss of generality we may assume that A is d–locally finite for all d. Let $\widetilde{v}$ be a base vertex of $N(\widetilde{A})$. Choose a path $\widetilde{\rho} \to \widetilde{X}$ beginning at $\widetilde{v}$ and ending with an edge dual to $\widetilde{B}$. Let v, ρ be the projections of $\widetilde{v}, \widetilde{\rho}$ to X. The elements of H_1H_2 are

represented by loops of the form $\alpha\rho\beta\rho^{-1}$, where α, β are closed paths in $N(A), N(B)$. We can also require that β does not have edges dual to B. Let $\gamma \to X$ be a closed path based at v representing an element outside H_1H_2. We want to find a finite cover $\widehat{X}$ of X, where the based lifts of γ and any path $\alpha\rho\beta\rho^{-1}$ as above have distinct endpoints. In other words, we want the based lift of $\gamma\rho$ and any of the lifts of ρ in the based elevation of $A \cup \rho$ to end with edges dual to distinct elevations of B. Suppose that ρ and $\gamma\rho$ have length $\leq d$. By Lemma 4.7 we can assume that A has injectivity radius $> 2d$. Then the quotient $A^{+d} = H_1 \backslash \widetilde{A}^{+d}$ embeds into X. Since A is d–locally finite, there are finitely many hyperplanes in A^{+d}. We now apply Theorem 4.2 to A^{+d}. Let $A^{+d} \to R(A^{+d})$ be the local isometry into the Salvetti complex $R(A^{+d})$ of the right-angled Artin group with generators corresponding to hyperplanes in A^{+d}. Apply Theorem 4.3 to the induced local isometry $R(A^{+d}) \to R(X)$. Consider its canonical completion $\widehat{R(X)} \to R(X)$ and the retraction $\widehat{R(X)} \to R(A^{+d})$. Take the pullback of the cover $\widehat{R(X)} \to R(X)$ to $\widehat{X} \to X$. We claim that $\widehat{X}$ is the required cover.

Indeed, the hyperplanes dual to the last edges of the lifts of $\gamma\rho$ and ρ starting at $N(A)$ are distinct in A^{+d}, hence their projections to $R(A^{+d})$ are also distinct. The retraction shows that the extensions of these projections are also distinct in $\widehat{R(X)}$. Hence the pullbacks of these extensions to $\widehat{X}$ are distinct as well. $\square$

Corollary 4.11. *Let X be a special cube complex with finitely many codim–2–hyperplanes. Let $A, B \subset X$ be two hyperplanes and let Q be a component of $A \cap B$. For each d there is a finite cover $\widehat{X} \to X$ with the following property. If elevations $\widehat{A}, \widehat{B}$ of $A, B \subset X$ intersect along an elevation of Q, then the entire $A \cap B$ projects to Q.*

Proof. In the proof of Lemma 4.10 we replace ρ with a vertex v in the carrier $N(Q)$. Choose a component Q' of $A \cap B$ distinct from Q, and a vertex v' in $N(Q')$. Consider the path $\gamma = \alpha\beta$ where α, β are starting at v and ending at v' in $N(A), N(B)$. In order for the retraction argument of Lemma 4.10 to go through for the element represented by γ it suffices that A is 0–locally finite. $\square$

Definition 4.12. Let $A \neq B$ be hyperplanes in a cube complex X and let $\mathcal{Q}$ be a family of components of $A \cap B$. Hyperplanes A, B have *double injectivity radius* $> d$ at $\mathcal{Q}$ if all the paths of length $\leq 2d$ in X starting at $N(A)$ and ending at $N(B)$ have the following property. They are path-homotopic to a concatenation at a vertex of $N(\mathcal{Q})$ of a pair of paths in $N(A)$ and $N(B)$. In particular $A \cap B = \mathcal{Q}$. In other words, if elevations $N(\widetilde{A}), N(\widetilde{B})$ of $N(A), N(B)$ to the universal cover of X are at distance $\leq 2d$, then $\widetilde{A} \cap \widetilde{B}$ is nonempty and projects to $\mathcal{Q}$.

Lemma 4.13. *Let X be a special cube complex with finitely many codim–2–hyperplanes. Let $A, B \subset X$ be two hyperplanes and let Q be a component of $A \cap B$. For each d there is a finite cover $\widehat{X} \to X$ with the following property. If elevations $\widehat{A}, \widehat{B} \subset \widehat{X}$ of A, B intersect along an elevation of Q, then they have double injectivity radius $> d$ at the full preimage of Q.*

Note that this property is preserved under passing to further covers. In particular, we can arrange that it holds for all A, B and Q simultaneously. Also note that some of the

components of the preimage of Q do not lie in $\widehat{A} \cap \widehat{B}$, so these will not be involved in the conclusion of Definition 4.12.

Proof. By Corollary 4.11 there is a finite cover $\widehat{X}$ of X with a nonempty intersection of elevations $\widehat{A} \cap \widehat{B}$ projecting to Q. Then passing to a regular cover and quotienting by the group permuting the components of $\widehat{A} \cap \widehat{B}$ reduces the situation to the case where $A \cap B$ is connected, i.e. $A \cap B = Q$.

Then once again let $X \to R$ be the local isometry into the Salvetti complex $R = R(X)$ of a finitely generated right-angled Artin group F, as in Theorem 4.2. Let $T_A, T_B \subset R$ denote the hyperplanes that are the images of A, B. Consider paths $\gamma \to R$ of length $\leq 2d$ starting at $N(T_A)$ and ending at $N(T_B)$ not path-homotopic to a concatenation of a pair of paths in $N(A), N(B)$. Since R is compact, there are finitely many such paths. Let $\mathcal{F}$ denote the family of conjugacy classes of elements of F determined by the closed paths $\alpha\beta\gamma^{-1}$, with α in $N(T_A)$ and β in $N(T_B)$. Then $\mathcal{F}$ is a union of classes determined by finitely many nontrivial double cosets of the form $H_1 H_2 g$, where $H_1 = \pi_1 T_A, H_2 = \pi_1 T_B$. Since double cosets of hyperplane subgroups in F are separable (particular case of Lemma 4.10, proved in [HW08, Cor 9.4]), the group F has a finite index subgroup $\widehat{F}$ disjoint from the set of elements whose classes lie in $\mathcal{F}$. Again let $\widehat{R} \to R$ be the finite cover corresponding to $\widehat{F} \subset F$ and let $\widehat{X} \to X$ be its pullback.

Once again $\widehat{X}$ is as required. The universal cover $\widetilde{X}$ of X is identified with a convex subcomplex of the universal cover $\widetilde{R}$ of R. Let $\widetilde{A}, \widetilde{B}$ be intersecting elevations of A, B in $\widetilde{X}$. Suppose $\widetilde{A}, g\widetilde{B}$ are at distance $\leq 2d$ for some $g \in \pi_1 \widehat{X} \subset \pi_1 \widehat{R}$. Hence the hyperplanes $\widetilde{T_A}, g\widetilde{T_B} \subset \widetilde{R}$ containing $\widetilde{A}, g\widetilde{B}$ intersect, by the large double injectivity radius of $\widehat{R}$. Since the convex hull of a pair points in intersecting hyperplanes contains an intersection point, hyperplanes $\widetilde{A}$ and $g\widetilde{B}$ intersect as well. □

5. Background on cubical small cancellation

In this section we review the main theorem of cubical small cancellation [Wis11]. It will be used in the proof of Theorem 2.4.

5.1. **Pieces.** Let X be a nonpositively curved cube complex. Let $\{Y_i \to X\}$ be a collection of local isometries of cube complexes. Then Y_i are nonpositively curved as well. The pair $\langle X | \{Y_i \to X\} \rangle$, or shortly $\langle X | Y_i \rangle$, is a *cubical presentation.* Its *group* is $\pi_1 X / \langle\langle \{\pi_1 Y_i\} \rangle\rangle$ and can be described as the fundamental group of the space X^* obtained from X by attaching cones along the Y_i. We denote by $\overline{X} = \langle\langle \{\pi_1 Y_i\} \rangle\rangle \backslash \widetilde{X}$ the cover of X in the universal cover $\widetilde{X}^*$ of X^*.

An *abstract (contiguous) cone-piece in* Y_i of Y_j is the intersection $P = \widetilde{Y}_i \cap \widetilde{Y}_j$ of some elevations $\widetilde{Y}_i, \widetilde{Y}_j$ of Y_i, Y_j to the universal cover $\widetilde{X}$ of X. Here we allow $i = j$, but in that case we require that the elevations are distinct in the sense that for the projections $P \to Y_i, Y_j$ there is no automorphism $Y_i \to Y_j$ such that the following diagram is commutative:

$$\begin{array}{ccc} P & \to & Y_i \\ \downarrow & \swarrow & \downarrow \\ Y_j & \to & X \end{array}$$

Note that an abstract cone-piece *in* Y_i actually lies in $\widetilde{Y}_i$.

Let $\widetilde{A}$ be a hyperplane in $\widetilde{X}$ disjoint from $\widetilde{Y}_i$. An *abstract (contiguous) wall-piece in* Y_i is the intersection $\widetilde{Y}_i \cap N(\widetilde{A})$.

A path $\alpha \to Y_i$ is a *piece in* Y_i, if it lifts to $\widetilde{Y}_i$ into an abstract piece in Y_i. We then denote by $|\alpha|_{Y_i}$ the combinatorial distance between the endpoints of a lift of α to $\widetilde{Y}_i$.

The cubical presentation $\langle X|Y_i\rangle$ satisfies the $C'(\frac{1}{n})$ *small-cancellation condition*, or shortly *is* $C'(\frac{1}{n})$, if $|\alpha|_{Y_i} < \frac{1}{n}\|Y_i\|$ for each piece α in Y_i. Recall that $\|Y_i\|$ denotes the minimum of the lengths of essential closed paths in Y_i.

5.2. **Ladder Theorem.** A *disc diagram* D is a compact contractible 2–complex with a fixed embedding in $\mathbb{R}^2$. Its *boundary path* $\partial_p D$ is the attaching map of the cell at ∞. The diagram is *spurless* if D does not have a *spur*, i.e. a vertex contained in only one edge. If X is a combinatorial complex, a *disc diagram in* X is a combinatorial map of a disc diagram into X.

Let $D \to \widetilde{X}^*$ be a disc diagram for a boundary path $\partial_p D \to \overline{X}$. Note that the 2–cells of $\widetilde{X}^*$ are squares or triangles, where the latter have exactly one vertex at a cone point. The triangles in D are grouped together according to these cone points into *cone-cells*. The *complexity* of D is the pair of numbers (# cone-cells of D, # squares of D), with lexicographic order.

In addition to spurs, there are two other types of positive curvature features at $\partial_p D$: *shells* and *cornsquares*. A cone-cell C adjacent to $\partial_p D$ is a *shell* if there is a path in $\partial C \cap \partial_p D$ (*outer path*) whose complement in ∂C (*inner path*) is a concatenation of ≤ 6 pieces. A pair of consecutive edges of $\partial_p D$ is a *cornsquare* if the carriers of their dual hyperplanes intersect at a square and surround a square subdiagram. A *ladder* is a disc diagram that is the concatenation of cone-cells and rectangles with cone-cells at extremities, as depicted in Figure 1. A single cone-cell is not a ladder. The following summarises the main results of cubical small cancellation theory

Theorem 5.1 ([Wis11, Thm 3.40])**.** *Assume* $\langle X|Y_i\rangle$ *is* $C'(\frac{1}{12})$. *Let* $D \to \widetilde{X}^*$ *be a minimal complexity disc diagram for a closed path* $\partial_p D \to \overline{X}$. *Then one of the following holds.*

- *D is a single vertex or a single cone-cell.*
- *D is a ladder.*
- *D has at least three spurs and/or shells and/or cornsquares. However, if there is no shell or spur, then there must be at least four cornsquares.*

5.3. **Small cancellation quotients.** We now prove that small cancellation quotients of special cube complexes have quasiconvex hyperplanes and allow separating elements from cosets and double cosets.

Lemma 5.2. *Let* $\langle X|Y_i\rangle$ *be a* $C'(\frac{1}{12})$ *cubical presentation. Suppose that each* Y_i *is virtually special with finitely many immersed hyperplanes. Let* $\widetilde{A} \subset \widetilde{X}$ *be a hyperplane and let* $g \in G - H$, *where* $G = \pi_1 X$ *and* $H = \mathrm{Stab}(\widetilde{A})$. *Then there are finite index subgroups* $P_i' \subset P_i = \pi_1 Y_i$ *such that:*

(1) The image $\overline{A}$ *of* $\widetilde{A}$ *in* $\overline{X} = \langle\langle \{P_i'\} \rangle\rangle \backslash \widetilde{X}$ *is quasi-isometrically embedded, and*

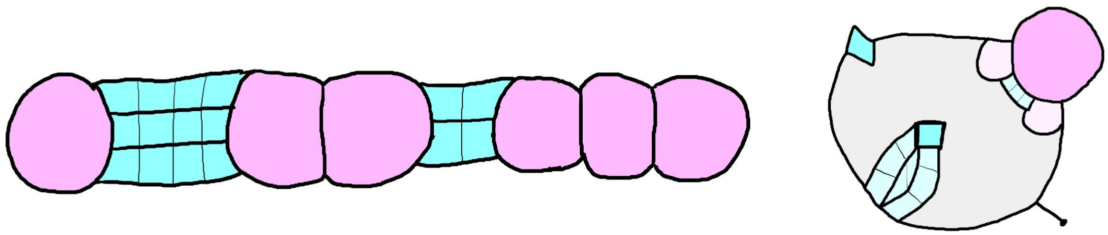

FIGURE 1. On the left is a ladder. On the right are two cornsquares, a spur, and a shell within a disc diagram.

(2) $\overline{g} \notin \overline{H}$ *in the quotient* $\overline{G} = G/\langle\langle\{P'_i\}\rangle\rangle$.

Proof. Let $12d$ be the maximum of $\|Y_i\|$. Then all the pieces have diameter $< d$. By Lemma 4.4 we can choose P'_i so that $\|Y'_i\| \geq 12d$ for all $Y'_i = P'_i \backslash \widetilde{Y}_i$. By Lemma 4.7, we can further choose P'_i so that every path in Y'_i of length $\leq 6d$ starting and ending at a hyperplane carrier N is path-homotopic into N. In other words, hyperplanes of Y'_i have injectivity radius $> 3d$. We also require that $P'_i \subset P_i$ are characteristic, so that $\langle X | Y'_i \rangle$ is $C'(\frac{1}{12})$.

We will show that the 1–skeleton of the carrier $N(\overline{A})$ of $\overline{A}$ is 2–quasiconvex in the 1–skeleton of $\overline{X}$. Let $D \to \widetilde{X}^*$ be a disc diagram bounded by a geodesic α in the 1–skeleton of $N(\overline{A})$ and a geodesic γ in the 1–skeleton of $\overline{X}$. We assume that D has minimal complexity among all disc diagrams with prescribed common endpoints of α and γ.

If in $\partial_p D = \alpha \cup \gamma$ there are two consecutive edges forming a cornsquare, they cannot both lie in α or both lie in γ: otherwise one could homotope D so that there is a square at that exact corner, and then push it out of the diagram to reduce the complexity. If there is a shell C in D whose outer path is contained in γ, then replacing the outer path by the inner path contradicts the fact that γ is a geodesic.

Finally, assume that the outer path O of C is contained in α. Let $\widetilde{Y}_i$ denote the universal cover of Y_i into which ∂C maps. Consider the copy of $\widetilde{Y}_i$ in $\widetilde{X}$ which contains a lift of O to $N(\widetilde{A})$. If $\widetilde{A}$ is disjoint from $\widetilde{Y}_i$, then O is a piece and ∂C is a concatenation of at most 7 pieces, which contradicts $\|Y'_i\| \geq 12d$. Otherwise put $\widetilde{A}_i = \widetilde{A} \cap \widetilde{Y}_i$. Hence O projects into the quotient $N(\overline{A}_i)$ of $N(\widetilde{A}_i)$ in Y'_i. Since the injectivity radius of the hyperplane $\overline{A}_i$ in Y'_i is $> 3d$, we can replace C by a square diagram which contradicts the minimal complexity of D.

Thus there can be at most two spurs and/or shells and/or cornsquares in D located where α and γ are concatenated. By Theorem 5.1, the disc diagram D is a single cone-cell or a ladder. For any of its cone-cells C denote $\alpha_C = \alpha \cap \partial C$, $\gamma_C = \gamma \cap \partial C$ and denote by λ_C, δ_C the remaining (possibly trivial) arcs of ∂C. Since λ_C, δ_C are pieces, $|\lambda_C|_{Y'_i}$ and $|\delta_C|_{Y'_i}$ (for appropriate i) are $< d$. Since $\|Y'_i\| \geq 12d$ and γ_C is a geodesic, by the triangle inequality $|\alpha_C|_{Y'_i} > 4d$. Then

$$|\gamma_C| = |\gamma_C|_{Y'_i} \geq |\alpha_C|_{Y'_i} - 2d > \frac{1}{2}|\alpha_C|_{Y'_i} = \frac{1}{2}|\alpha_C|.$$

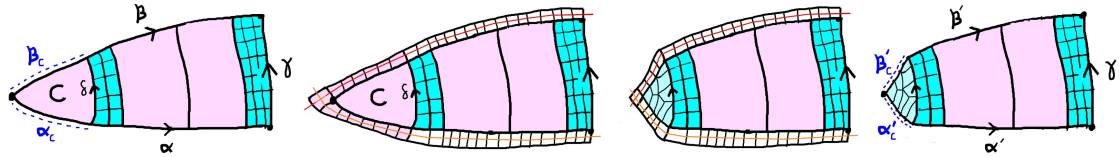

FIGURE 2. The shell C in the first diagram is surrounded by two hyperplanes and a short inner path δ as in the second diagram, it therefore can be replaced by a square diagram with $\alpha \cap \partial C, \beta \cap \partial C$ cropped, to obtain a smaller complexity counterexample as in the last diagram.

Thus we have $|\gamma| \geq \frac{1}{2}|\alpha|$. This proves assertion (1).

For assertion (2) assume additionally that d is chosen to be greater than the length of a shortest closed path γ in X representing g based at a vertex of $N(A)$. If $\overline{g}$ lies in $\overline{H}$, then the lift of γ to $\overline{X}$ forms a loop with a path α in $N(\overline{A})$. We form the disc diagram D as above. If it contains a cone-cell C, then the length of $\gamma_C \subset \gamma$ is at least $2d$ which contradicts the choice of d. Otherwise D is a path and γ lies in $N(\overline{A})$, hence $g \in H$, contradiction. □

The following clarifies and generalises [Wis11, Thm 16.23].

Lemma 5.3. *Let $\langle X|Y_i\rangle$ be a $C'(\frac{1}{12})$ cubical presentation. Suppose that each Y_i is virtually special with finitely many immersed codim–2–hyperplanes. Let $H_1, H_2 \subset G = \pi_1 X$ be stabilisers of intersecting hyperplanes $\widetilde{A}, \widetilde{B} \subset \widetilde{X}$ and let $g \in G - H_1H_2$. There are finite index subgroups $P_i' \subset P_i = \pi_1 Y_i$ such that $\overline{g} \notin \overline{H}_1\overline{H}_2$ in the quotient $\overline{G} = G/\langle\langle\{P_i'\}\rangle\rangle$.*

Proof. We keep the notation from Lemma 5.2. Let $\widetilde{Q} = \widetilde{A} \cap \widetilde{B}$ and denote its projections to $\overline{X}, X$ by $\overline{Q}, Q$. We choose d so that:

- pieces have diameter $< d$, and
- any path γ in X representing g and based at a vertex of $N(Q)$ has length $< d$.

By Lemmas 4.4, 4.7, and 4.13 we can choose P_i' characteristic in P_i, such that $Y_i' = P_i' \backslash \widetilde{Y}_i$ satisfy:

- $\|Y_i'\| \geq 12d$,
- all hyperplanes in Y_i' have injectivity radius $> 4d$, and
- all pairs of hyperplanes in Y_i' have double injectivity radius $> 3d$ at the P_i/P_i' orbit of their intersection.

We now argue by contradiction to prove the lemma. If $\overline{g}$ lies in $\overline{H}_1\overline{H}_2$, then there is a disc diagram $D \to \widetilde{X}^*$ bounded by a closed path $\alpha\gamma\beta^{-1}$, where α, β^{-1} are paths in $N(\overline{A}), N(\overline{B})$ concatenated at a vertex $\overline{v} \in N(\overline{Q})$, and γ projects to a shortest closed path based at $N(Q)$ representing g. Assume that D has minimal number of spurs among minimal complexity diagrams. Then α and β do not share their first edge.

An outer path of a shell cannot be contained entirely in α, β or γ, as before. Moreover, it cannot be contained in $\alpha\gamma$ (or $\gamma\beta^{-1}$), since the length of γ is $< d$ and the length of the inner path is $< 6d$: indeed, if the intersection of the outer path with α is a piece, then

this contradicts $\|Y_i'\| \geq 12d$, and otherwise this contradicts minimal complexity since the hyperplane injectivity radius is $> 4d$.

Let D_o be the spurless disc diagram obtained from D by removing the possible chains of spurs at $\alpha \cap \gamma$ and $\gamma \cap \beta^{-1}$. Since $\langle X | Y_i' \rangle$ is $C'(\frac{1}{12})$, by Theorem 5.1 the disc diagram D_o must have a shell C at $\overline{v}$. Let α_C and β_C equal the subpaths $\alpha \cap \partial C$ and $\beta \cap \partial C$. Suppose ∂C maps to Y_i'.

Choose elevations $\widetilde{Q} \subset \widetilde{A}, \widetilde{B} \subset \widetilde{X}$ and lift $\overline{v}$ to $\widetilde{v} \in N(\widetilde{Q})$. Let $\widetilde{Y}_i$ be the elevation of Y_i at $\widetilde{v}$. Let $\widetilde{A}_i = \widetilde{A} \cap \widetilde{Y}_i$, $\widetilde{B}_i = \widetilde{B} \cap \widetilde{Y}_i$. If both $\widetilde{A}_i, \widetilde{B}_i$ are empty, then both α_C, β_C are pieces which contradicts $\|Y_i'\| > 12d$. If exactly one of $\widetilde{A}_i, \widetilde{B}_i$ is empty, say $\widetilde{A}_i$, then α_C is a piece. This contradicts the fact that the injectivity radius of $\widetilde{B}_i$ is $> 4d$. Hence each $\widetilde{A}_i, \widetilde{B}_i$ is nonempty and D shows that they intersect in nonempty $\widetilde{Q} \cap \widetilde{Y}_i$.

The complement δ in ∂C of $\beta_C^{-1}\alpha_C$ is an inner path of the shell C, hence a concatenation of at most 6 pieces. The double injectivity radius in Y_i' is $> 6d$. Hence δ is homotopic in Y_i' to a concatenation at $pN(\overline{Q})$ of paths α_C', β_C' in $N(\overline{A}_i), N(\overline{B}_i)$ for some $p \in P_i/P_i'$. In other words, there exists a square diagram with boundary $\alpha_C'\delta{\beta'}_C^{-1}$. We replace C by this square diagram, and replace the subpath α_C of α by α_C' to obtain α', and similarly obtain β'. Translating the whole diagram by p^{-1} gives us a disc diagram at $N(\overline{Q})$ with the boundary decomposition analogous to that of D but a smaller number of cone-cells, which contradicts the minimal complexity assumption. See Figure 2. $\square$

6. Specialisation

In this section we prove Theorem 2.4 (Specialisation). We will need the following version of the Special Quotient Theorem.

Theorem 6.1 ([Wis11, Lem 16.13])**.** *Let G be the fundamental group of a sparse virtually special cube complex. Suppose that G is hyperbolic relative to virtually abelian subgroups $\{E_n\}$ stabilising the quasiflats. There are finite index subgroups $E_n^\circ \subset E_n$ such that for any further finite index subgroups $E_n^c \subset E_n^\circ$ the quotient $G/\langle\langle\{E_n^c\}\rangle\rangle$ is hyperbolic and virtually compact special. Moreover, each E_n/E_n^c embeds into $G/\langle\langle\{E_n^c\}\rangle\rangle$.*

Proof of Theorem 2.4. To prove that the action of G on $\widetilde{X}$ is virtually special, we will verify the conditions of Criterion 2.3. Freeness and finiteness Conditions (1)–(3) of Criterion 2.3 are Hypothesis (i) of Theorem 2.4. We now verify Condition (4) of Criterion 2.3. Let H be the stabiliser of a hyperplane $\widetilde{A} \subset \widetilde{X}$. Let $g \in G - H$. We will find finite index subgroups $P_i' \subset P_i$ such that:

(a) $\overline{G} = G/\langle\langle\{P_i'\}\rangle\rangle$ is virtually compact special hyperbolic,
(b) the image $\overline{H}$ is quasiconvex in $\overline{G}$,
(c) $\overline{g} \notin \overline{H}$.

The result then follows from separability of quasiconvex subgroups in hyperbolic virtually compact special groups [HW08, Thm 7.3].

By Hypothesis (iv) and Theorem 6.1, there are $E_n^\circ \subset E_n$ such that $P_i' \cap E_n \subset E_n^\circ$ implies that $\overline{G}$ splits as a graph of virtually compact special hyperbolic groups with finite edge groups. Then $\overline{G}$ is virtually compact special hyperbolic and Condition (a) is

satisfied. By Hypothesis (ii), there are indeed finite index subgroups $P_i' \subset P_i$ satisfying $P_i' \cap E_n \subset E_n^\circ$.

In order to arrange Conditions (b) and (c) we appeal to cubical small cancelation theory. The group $\overline{G}$ is the fundamental group of the complex X^* obtained from $G\backslash\widetilde{X}$ by attaching cones C_i along immersed subcomplexes $Y_i' = P_i'\backslash\widetilde{Y}_i$. By Hypothesis (iii), the complexes Y_i' are virtually special and have finitely many immersed codim–2–hyperplanes.

Since $\widetilde{Y}_i$ are superconvex, there is a constant d_w that bounds the diameter of contiguous wall-pieces. By relative hyperbolicity and relative cocompactness there is a constant d_c that bounds the diameter of contiguous cone-pieces. Let $d = \max\{d_w, d_c\}$. By Lemma 4.4, we can assume $\|Y_i'\| > 12d$. Hence if P_i' are normal in P_i, then $\langle X|Y_i'\rangle$ is $C'(\frac{1}{12})$.

We can then apply Lemma 5.2. Condition (c) follows directly from Lemma 5.2(2). By Lemma 5.2(1) the quotient $\overline{A}$ of A in $\overline{X} = \langle\langle\{P_i'\}\rangle\rangle\backslash\widetilde{X}$ is quasi-isometrically embedded. Let $K \subset \widetilde{X}$ be the subcomplex from the definition of relatively cocompact action of G on $\widetilde{X}$. Then $\overline{G}$ acts cocompactly on $\langle\langle\{P_i'\}\rangle\rangle\backslash GK$, which we will denote by $\overline{X}_c$. Moreover, each component of $\overline{X} - \overline{X_c}$ intersects $\overline{X}_c$ along a uniformly bounded subcomplex, since P_i' is of finite index in P_i and P_i' acts cocompactly on $\widetilde{Y}_i \cap GK$. Hence $\overline{A} \cap \overline{X}_c$ is quasi-isometrically embedded in $\overline{X}_c$, which means that its stabiliser $\overline{H}$ is quasiconvex in $\overline{G}$, as desired. This completes the proof of Condition (4) of Criterion 2.3.

In order to prove Condition (5) of Criterion 2.3 we need to strengthen (c) to

(c′): $\overline{g} \notin \overline{H}_1\overline{H}_2$,

where H_1, H_2 are the stabilisers of hyperplanes $\widetilde{A}_1, \widetilde{A}_2 \subset \widetilde{X}$ and $g \in G - H_1H_2$. This can be arranged using Lemma 5.3. Once we have (a), (b) and (c′) we appeal to [Min06, Thm 1.1], which says that in hyperbolic groups with separable quasiconvex subgroups, double cosets of quasiconvex subgroups are separable as well. □

References

[Ago04] Ian Agol, *Tameness of hyperbolic 3-manifolds* (2004), available at `arXix:GT/0405568`.

[Ago08] ———, *Criteria for virtual fibering*, J. Topol. **1** (2008), no. 2, 269–284.

[Ago12] ———, *The virtual Haken conjecture*, with an Appendix by Ian Agol, Daniel Groves, and Jason Manning (2012), preprint, available at `arXiv:1204.2810`.

[AFW12] Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton, 3*–manifold groups* (2012), preprint, available at `arXiv:1205.0202`.

[BW11] Hadi Bigdely and Daniel T. Wise, *Quasiconvexity and relatively hyperbolic groups that split* (2011), preprint.

[Bri01] Martin R. Bridson, *On the subgroups of semihyperbolic groups*, Monographie de L'Enseignement Mathématique **38** (2001), 85–111.

[Can96] Richard D. Canary, *A covering theorem for hyperbolic* 3*-manifolds and its applications*, Topology **35** (1996), no. 3, 751–778.

[CG06] Danny Calegari and David Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **19** (2006), no. 2, 385–446.

[DJ00] Michael W. Davis and Tadeusz Januszkiewicz, *Right-angled Artin groups are commensurable with right-angled Coxeter groups*, J. Pure Appl. Algebra **153** (2000), no. 3, 229–235.

[HW08] Frédéric Haglund and Daniel T. Wise, *Special cube complexes*, Geom. Funct. Anal. **17** (2008), no. 5, 1551–1620.

[HW10] ———, *Coxeter groups are virtually special*, Adv. Math. **224** (2010), no. 5, 1890–1903.

[Hru10] G. Christopher Hruska, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. **10** (2010), no. 3, 1807–1856.

[HW11] Christopher Hruska and Daniel T. Wise, *Finiteness properties of cubulated groups* (2011), submitted, available at `http://www.math.mcgill.ca/wise/papers.html`.

[HW99] Tim Hsu and Daniel T. Wise, *On linear and residual properties of graph products*, Michigan Math. J. **46** (1999), no. 2, 251–259.

[KM04] P. B. Kronheimer and T. S. Mrowka, *Dehn surgery, the fundamental group and* $SU(2)$, Math. Res. Lett. **11** (2004), no. 5-6, 741–754.

[Lee95] Bernhard Leeb, 3-*manifolds with(out) metrics of nonpositive curvature*, Invent. Math. **122** (1995), no. 2, 277–289.

[Liu11] Yi Liu, *Virtual cubulation of nonpositively curved graph manifolds* (2011), available at `arXiv:1110.1940`.

[MPW11] Eduardo Martínez-Pedroza and Daniel T. Wise, *Relative quasiconvexity using fine hyperbolic graphs*, Algebr. Geom. Topol. **11** (2011), no. 1, 477–501.

[Min06] Ashot Minasyan, *Separable subsets of GFERF negatively curved groups*, J. Algebra **304** (2006), no. 2, 1090–1100.

[PW11] Piotr Przytycki and Daniel T. Wise, *Graph manifolds with boundary are virtually special* (2011), submitted, available at `arXiv:1110.3513`.

[RW98] J. Hyam Rubinstein and Shicheng Wang, π_1-*injective surfaces in graph manifolds*, Comment. Math. Helv. **73** (1998), no. 4, 499–515.

[SW11] Michah Sageev and Daniel T. Wise, *Periodic flats in* CAT(0) *cube complexes*, Algebr. Geom. Topol. **11** (2011), no. 3, 1793–1820.

[Thu80] William P. Thurston, *The Geometry and Topology of Three-Manifolds* (1980), Princeton University course notes, available at `http://www.msri.org/publications/books/gt3m/`.

[Wis06] Daniel T. Wise, *Subgroup separability of the figure 8 knot group*, Topology **45** (2006), no. 3, 421–463.

[Wis11] ______, *The structure of groups with quasiconvex hierachy* (2011), available at `http://www.math.mcgill.ca/wise/papers.html`.

Inst. of Math., Polish Academy of Sciences, Śniadeckich 8, 00-956 Warsaw, Poland
E-mail address: `pprzytyc@mimuw.edu.pl`

Math. & Stats., McGill University, Montreal, Quebec, Canada H3A 2K6
E-mail address: `wise@math.mcgill.ca`